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Five-branes, seven-branes and five-dimensional E_n field theories

Oliver DeWolfe, Amihay Hanany, Amer Iqbal and Emanuel Katz

Center for Theoretical Physics, Laboratory for Nuclear Science,
Department of Physics, Massachusetts Institute of Technology
Cambridge, Massachusetts 02139, U.S.A.
E-mail: odewolfe@mit.edu, hanany@mit.edu, amikatz@mit.edu,
iqbal@ctpa04.mit.edu

Abstract: We generalize the (p, q) 5-brane web construction of five-dimensional field theories by introducing (p, q) 7-branes, and apply this construction to theories with a one-dimensional Coulomb branch. The 7-branes render the exceptional global symmetry of these theories manifest. Additionally, 7-branes allow the construction of all E_n theories up to $n = 8$, previously not possible in 5-brane configurations. The exceptional global symmetry in the field theory is a subalgebra of an affine symmetry on the 7-branes, which is necessary for the existence of the system. We explicitly determine the quantum numbers of the BPS states of all E_n theories using two simple geometrical constraints.

Keywords: Field Theories in Higher Dimensions, Brane Dynamics in Gauge Theories, Global Symmetries.

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Contents

1. Introduction

Five-dimensional $\mathcal{N} = 1$ gauge theories have been much explored in the last few years. Those theories with a one-dimensional Coulomb branch have a particularly rich variety of descriptions in string theory. They were first described by the dynamics of D4-branes in a Type I$'$ background of D8-branes and orientifold O8-planes, where it was determined that they possess enhanced exceptional global symmetry at the origin of parameter space, which is a superconformal fixed point [1]. These theories also appear in M-Theory compactifications on a Calabi-Yau threefold with a vanishing four-cycle [2, 3]. Such complex surfaces, called del Pezzo manifolds, are in one-to-one correspondence with the five-dimensional theories with a single Coulomb modulus. The BPS spectrum can be obtained by using the intersection properties of various curves in the del Pezzo [4, 5].

Finally, some of the five-dimensional theories appear as the effective field theory on a web of (p, q) 5-branes in Type IIB [6, 7, 8, 9]. The web has a single face, whose size corresponds to the parameter of the Coulomb branch. BPS states come from (p, q) strings and string junctions stretched between the 5-branes. This realization is appealing, as parameters (such as masses and the gauge coupling) and moduli of the theory may be easily read off the 5-brane geometry. However, it has its drawbacks. The exceptional global symmetry is not particularly evident, and must be inferred from the correspondence with other presentations. Moreover, this setup becomes problematic beyond the E_3 configuration. Those brane configurations producing E_4 and E_5 involve parallel external legs, which could result in six-dimensional modes. Those corresponding to E_6, E_7 and E_8 are worse, for the external legs would cross, and the configurations cannot exist. This is related to the lack of a toric description for the corresponding del Pezzos [10].

In this paper we generalize the (p, q) 5-brane setup so as to resolve these problems. We do this by introducing (p, q) 7-branes to the picture. This can be done at no additional cost in supersymmetry. The external (p, q) 5-branes are allowed to terminate at finite distance on the appropriate (p, q) 7-brane. Motion of each 7-brane along the direction of the 5-brane does not affect the five-dimensional theory. One may then bring the 7-branes within the 5-brane face, so that the external legs vanish altogether. The BPS states of the theory are hypothesized to switch from being 5-5 string junctions to 5-7 junctions for certain regions of parameter space.

From this point of view the global symmetries are manifest, as the technology of enhanced 7-brane algebras can be brought to bear [11, 12, 13, 14, 15, 16, 17]. The symmetry algebras that appear on 7-brane configurations have been classified [16, 17], and it is possible to determine the weight vector associated to any string junction with support on such a configuration [13]. Moreover, the string junctions have a natural intersection form inherited from curves in K3 [14, 18], which may be used to determine the BPS spectrum in a fashion similar to the del Pezzo case. In fact, the symmetries of the 7-branes turn out to be not just E_n, but the affine algebras $\widehat{E}_n$, first observed in [15]. We will show how the constraint that a junction end on the 5-branes reduces the effective symmetry to just E_n, consistent with our expectations for the field theory. The affine character of the 7-branes is not irrelevant, however; the 5-branes themselves have an associated weight vector corresponding to the imaginary root of the affine algebra, and in fact a 7-brane configuration must be affine in order to support a 5-brane web with a single face.

Moreover, with both 5-branes and 7-branes we can describe any of the E_n theories, all the way up to E_8. This is because there is no obstruction to placing certain additional 7-branes inside the 5-brane web, thereby forming the entire E_n series. Then for $n \geq 4$, these 7-branes simply cannot be brought out of the 5-brane web and moved off to infinite distance without creating the problem of parallel/crossing legs. When the face shrinks to zero size, we find that for $n \leq 3$ the configuration reverts to having no 7-branes within the face before reaching the superconformal point. For precisely the $n \geq 4$ cases where no configuration existed without 7-branes, the fixed point occurs when the face collapses around a subset of the 7-branes. The finite algebra $E_n \subset \widehat{E}_n$ is restored on this subset of branes.

In the section 2 we will recapitulate the (p,q) 5-brane web setup and the nature of the five-dimensional gauge theories. Then in section 3 we discuss placing 7-branes in the background and allowing 5-brane external legs to end on them, and section 4 describes how when the 7-branes are moved inside the face, the 5-5 strings can be replaced by 5-7 strings. Section 5 reviews the way string junctions realize algebras on 7-branes, and shows how the E_n theories correspond to $\widehat{E}_n$ brane configurations. The fact that only a finite symmetry is realized on the spectrum is explained. In section 6 we detail the case of E_1 explicitly. We then compare 5-7-brane techniques to the del Pezzo realization of five-dimensional theories and explore the predictions for BPS spectra in section 7. Finally section 8 discusses the approach to the superconformal point. An appendix discusses relevant 7-brane computations.

2. Five-dimensional theories and (p,q) 5-brane webs

We are concerning ourselves with the study of five-dimensional gauge theories with 8 supercharges. The vector multiplets contain one real scalar, and we are interested in those theories with a one-dimensional Coulomb branch associated to a single vector multiplet. We denote the various theories by their associated exceptional global symmetry, E_n. For a discussion of more general five-dimensional theories with 8 supercharges, see [19, 20].

Most of these theories, the ones with $1 \leq n \leq 8$, can be realized as the low-energy limit of SU(2) with $N_f = n-1$ hypermultiplets. The exceptional symmetries were found by Seiberg [1] using Type I′ arguments. The restriction $N_f \leq 7$ was motivated from field theory as well. Two additional theories, $\tilde{E}_1$ and E_0, were identified by flowing from the E_2 theory [3]. The former has a $u(1)$ global symmetry, and is identified as the IR limit of a pure SU(2) gauge theory with nontrivial $\mathbb{Z}_2$ θ-angle [2], while the latter has no global symmetry and no interpretation as a gauge theory since there are no W-bosons on the Coulomb branch.

Five-dimensional gauge theories are characterized by a gauge coupling $1/g^2$ with the dimensions of mass, as well as (real) masses for possible hypermultiplet matter. The prepotential is restricted to be cubic. Even in the absence of a cubic term classically, one is generically generated at one-loop. A classical cubic term is an additional parameter, and can be thought of as having arisen from matter already integrated out. The BPS spectrum includes not just W-bosons and quarks, but also instantons, which are particles in 5D. Furthermore, the BPS spectrum also features magnetic strings.

Let us review the (p, q) 5-brane setup. The (p, q) 5-branes fill the common directions 01234, which is the spacetime of the effective five-dimensional theory. In addition each 5-brane occupies 1-dimensional support in the 56 plane, with the slope of the curve determined at each point by the charges (p, q) of the 5-brane and the expectation value of the complexified coupling $\tau = \chi + ie^{-\phi}$:

$$\Delta x_5 + i\Delta x_6 \,||\, p + \tau q\,. \tag{2.1}$$

The 5-branes occupy a point in the 789 directions, which will not play a role in what follows. Three 5-branes can meet at a point as long as charge is conserved: $\sum_i (p_i, q_i) = 0$. Thus we obtain a (p, q) 5-brane web.

The slope condition stems from the requirement of supersymmetry. (Together with charge conservation, it can also be thought of as the condition for mechanical equilibrium.) Type IIB string theory has 32 supercharges, organized into two 16-component Majorana-Weyl spinors Q_L, Q_R with the same chirality: $\overline{\Gamma} Q_L = Q_L$, $\overline{\Gamma} Q_R = Q_R$, where $\overline{\Gamma} = \Gamma_0, \ldots, \Gamma_9$. If we place a D5-brane (= (1,0)-brane) in the 012345 directions, the supersymmetries preserved are $\epsilon^L Q_L + \epsilon^R Q_R$ with the constant spinors ϵ satisfying

$$\epsilon^L = \Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\epsilon^R\,, \tag{2.2}$$

which as usual preserves 1/2 of the supersymmetry of the vacuum. If in addition we place an NS 5-brane (= $(0, 1)$-brane) in the 012346 directions, we have the additional constraints

$$\begin{aligned} \epsilon^L &= \Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_6\epsilon^L\,, \\ \epsilon^R &= -\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_6\epsilon^R\,, \end{aligned} \tag{2.3}$$

and a total of 1/4 of the supersymmetry is preserved. If any (p, q) 5-brane is added to the configuration, no additional supersymmetry will be broken as long as it fills the 01234 directions and a line in the 56 plane determined by (2.1). Thus the (p, q) 5-brane web has an effective five-dimensional field theory with 8 supercharges, or $\mathcal{N} = 1$. Each face of the 5-brane web corresponds to a modulus on the Coulomb branch, and thus we will be considering configurations with one face.

The BPS particles in the spectrum arise from strings and string junctions stretching between the 5-branes, while the BPS strings are 3-branes wrapping the face. IIB strings are also characterized by charges (p, q), and they form junctions and webs in the plane as 5-branes do. A (p, q) string can end on a (p, q) 5-brane, and they must be perpendicular at the contact point; thus the slopes of string segments are rotated 90 degrees from those of 5-branes (2.1).

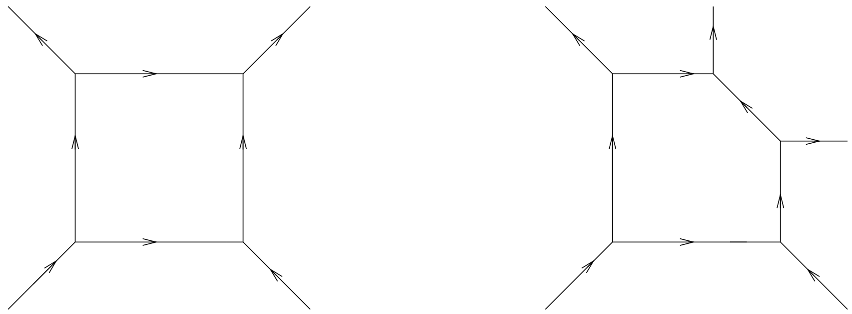

Figure 1: *a*) The 5-brane web corresponding to the E_1 theory. *b*) The 5-brane web corresponding to the E_2 theory, with horizontal external leg (D5-brane) generating a flavor.

In fig. 1*a*, we show the 5-brane web for the E_1 theory, with the value $\tau = i$ chosen to make the slopes of the 5-branes agree with their charge vectors; in the absence of 7-branes τ is a constant, and its value is a superfluous parameter from the point of view of the five-dimensional theory [6, 7]. The face is made of two D5-branes and two NS5-branes; the external legs have charges $(1, 1)$ and $(1, -1)$. The basic BPS states are the W-boson, realized as a F-string connecting the D5-branes, and the instanton, a D-string stretching between NS5-branes; these are a doublet of the $E_1 = \mathrm{SU}(2)$ global symmetry, and other BPS states can be thought of as linear combinations of these. This field theory is pure SU(2) gauge theory (with vanishing θ-angle); flavors are associated with D5-brane external legs, as in the realization of E_2 in fig. 1*b*.

Let us demonstrate that 7-branes may be added to the configuration without breaking any more supersymmetry. Combining (2.2) and (2.3), we obtain

$$\begin{aligned} \epsilon^L &= -(\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5)(\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_6)\epsilon^R \\ &= -\Gamma_5\Gamma_6\epsilon^R , \end{aligned} \tag{2.4}$$

but since $\overline{\Gamma} Q_{L,R} = Q_{L,R}$ and $\overline{\Gamma}^2 = 1$, we have $\overline{\Gamma}\epsilon^{L,R} = \epsilon^{L,R}$ as well, and then

$$\epsilon^L = -\Gamma_5\Gamma_6\overline{\Gamma}\epsilon^R \tag{2.5}$$

$$= \Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_7\Gamma_8\Gamma_9\epsilon^R , \tag{2.6}$$

which is recognized as the constraint on the preserved supersymmetries in the presence of a D7-brane at a point in the 56 plane and filling the other directions. In fact, all (p, q) 7-branes preserve this same supersymmetry. Thus we can introduce 7-branes to the 5-brane web without breaking any additional supersymmetry. This is reminiscent of the introduction of D3-branes to a different D5/NS5 system with a three-dimensional effective field theory [21]. Since 7-branes are a source for τ the condition (2.1) will become a complicated function of position in the 56 plane.

3. 7-branes and 5-brane webs

3.1 Ending the 5-brane web on 7-branes

Consider the E_1 web configuration with a D7-brane inserted; we have placed the 7-brane branch cut where it will not affect the 5-brane system, as shown in fig. $2a$. Fundamental strings can stretch between the D7-brane and the D5-branes, creating quark hypermultiplets in the fundamental representation of SU(2), the gauge group of the 5D theory. The length of these strings corresponds to the mass of the quarks, and thus the vertical position of the D7 is the quark bare mass parameter. Moving the D7 brane horizontally within the face does not change the length of the strings, and therefore does not affect any parameters of the theory [21].

In fact, one can even move the D7 brane horizontally past the NS5 brane. Via the familiar brane creation [21], a D5-brane prong will form, going to the right of the NS5-brane and ending on the D7-brane, as in fig. $2b$. The quarks now result from strings going between the original D5-branes in the face and the newly-created D5, and as we move the D7-brane horizontally, the quark length and hence the mass parameters do not change. Thus horizontal motion of the D7 is irrelevant to the field theory everywhere. Since we can move the D7-brane to infinity, we conclude that from the point of view of the 5D theory a semi-infinite D5-brane is equivalent to a finite D5-brane ending on a D7. This is consistent with the fact that both the D7 brane and the semi-infinite D5 brane are infinitely massive from the 5D perspective. This makes their vertical motion a parameter of the theory.

One can tune the locations of the external legs such that as the face shrinks, the NS5-branes collapse together before the D5-branes do. In this case the situation can be interpreted as an SU(2) living on the NS5-branes, rather than the D5-branes, thanks to S-duality; this is known as continuation past infinite coupling [6, 7]. Quarks in the NS5 theory can be provided by external NS5 legs, which we can analogously end on $(0, 1)$ 7-branes, again expecting that motion of the 7-brane along the path of the NS5 will not affect the parameters of the 5D theory; these parameters are related to the parameters and moduli of the theory before continuation past infinite

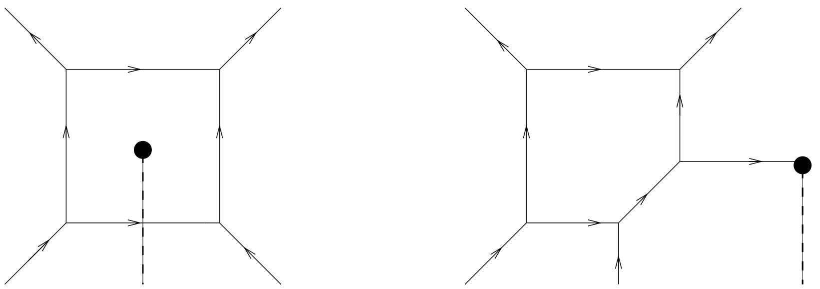

Figure 2: A D7-brane added to the E_1 configuration, adding a flavor and producing E_2.

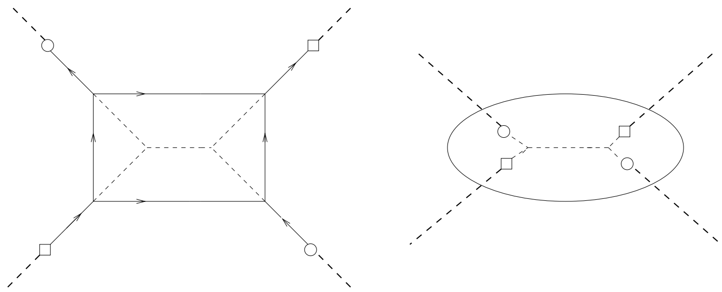

Figure 3: *a*) The external 5-branes of the E_1 web can end on 7-branes. *b*) The 7-branes move within the 5-brane face, removing the external 5-branes entirely.

coupling. Thus in general, we may take both NS5 and D5-branes to end on $(0,1)$ and $(0,1)$ 7-branes respectively, and motion along the external legs should not affect any parameters or moduli of the D5 brane theory.

Since this is true for both types of 7-branes, we propose that any semi-infinite (p,q) 5-brane leaving the (p,q) web may be replaced with a finite 5-brane ending on the appropriate (p,q) 7-brane. This 7-brane can be moved along its prong without changing the 5D field theory. For example, the pure SU(2) theory configuration can be modified so that all external legs end on 7-branes, as in fig. 3*a*. The dashed lines represent the 7-brane branch cuts, which point away from the web so as not to disturb the charges of the 5-branes. The dotted lines show the web at the origin of the Coulomb branch. Note that the presence of 7-branes creates a non-trivial metric. Thus the 5-branes follow curved geodesics, rather than straight lines of a given slope, and we will use the slopes merely to indicate the 5-brane charges.

Let us now ask what would happen if we move all the 7-branes, along their prong directions, into the center of the web. As each brane passes the corner to which it is connected, brane creation will occur and the 5-brane prong will disappear. The result will be a single 5-brane loop going around the 7-branes, with its (p,q) nature changing as it passes the various branch cuts, as in fig. 3*b*. Since the loop is closed, this is indeed a five-dimensional theory.

Evidence for the existence of such a loop may be found by taking the parameter $1/g^2$ to be very large, resulting in a short and wide web, and then taking the size of the face sufficiently large. The four 7-branes become two $(1,-1)$ and $(1,1)$ pairs, each of which is the quantum resolution of the O7 orientifold [22]. For sufficiently large modulus they can be treated as two O7-planes, and in this limit the geodesic corresponding to the loop may be explicitly constructed. Indeed, its length is found to be independent of the size of the face, as expected from a modulus deformation of the web. In addition, performing a T-duality transformation along the 5-brane direction, one obtains Seiberg's IIA setup of the same gauge theory. Here, the D5 brane becomes a D4 brane, and the two O7-planes merge into a single O8-plane.

3.2 BPS states as 5-7 strings

In the case of the familiar transition of a pronged D7 brane across an NS5 brane, it is known that certain states of the 5D theory (namely the quarks) which came from D5-D5 strings, instead originate from D5-D7 strings. We would like to show that upon moving the 7-branes into the face of the SU(2) web, every BPS state described by 5-5 strings may be alternatively described by 5-7 strings. Here, a description entails understanding of a state's quantum numbers under the local and global symmetries. Since the states we consider are BPS, this information is also enough to proscribe their masses.

The map between 5-5 and 5-7 strings is achieved through the continuous deformation of the 5-5 string junctions. Once the 7-branes are inside the face, the web consists of a single 5-brane loop. Hence, one may drag the endpoints of each 5-string junction until they lie, for example, entirely on the top (D5-brane portion) of the loop. As the junction is deformed in such fashion, prongs on 7-branes will form via brane creation. The result is a 5-7 string junction, starting at the 7-branes and ending on the top of the 5-brane loop. For the cases of the instanton and the W-boson this process is displayed in fig. 4.

Let us stress that metric around the 7-branes is nontrivial, and therefore for generic locations of the 7-branes we do not know if the actual BPS (geodesic) junction is a 5-5 or a 5-7 string. Rather, we do not expect these deformations to change the quantum numbers associated to the BPS state. One way to see this is to note that Type IIB-theory with 7-branes on a sphere is dual to M-theory on a K3-surface, and string junctions correspond to membranes wrapping complex curves in K3. The quantum numbers of a junction are determined by the homology cycle wrapped by

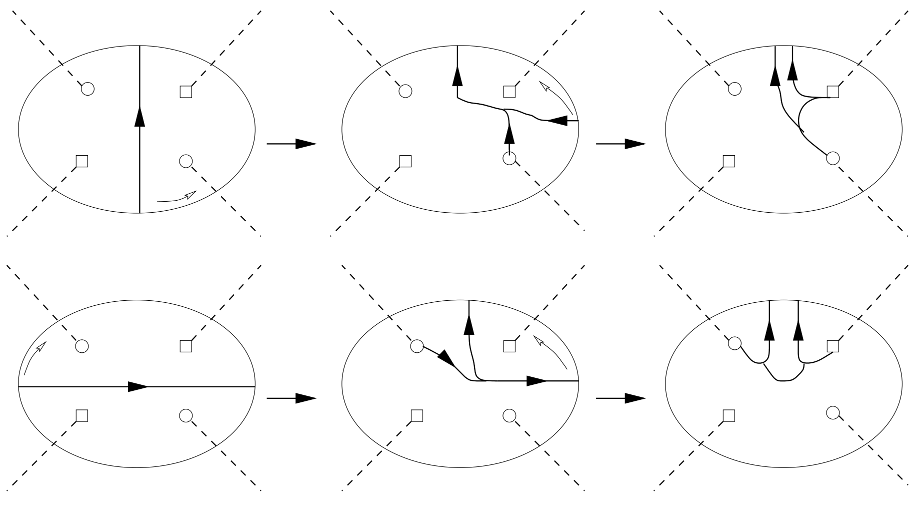

Figure 4: The W-boson (top) and instanton (bottom) can be continuously deformed from being 5-5 stings to 5-7 string junctions.

the membrane and its intersection numbers, which are invariant under continuous transformations. Hence, to describe the junction curve we may choose any convenient homology representative, whether it is the BPS (area minimizing) one or not.

The advantage of representing the states of the theory as 5-7 string junctions is that the action of the global symmetry becomes more transparent. Each 7-brane has a U(1) gauge theory living on its worldvolume. When 7-branes of various types are able to become coincident, gauge symmetry enhancement will occur in their worldvolume theory. The string junctions leaving the 7-branes will thus fall naturally under representations of this enhanced symmetry group. Therefore, this enhancement translates into a global symmetry in the 5D theory. This is the subject of the next section.

4. String junctions and 7-branes

We now turn to a review of the symmetries arising on (p,q) 7-brane configurations, and how they are realized on (p,q) strings ending on the 7-branes. Using the notation of [11], we will often refer to 7-branes with charges $[1,0]$, $[1,-1]$ and $[1,1]$ as **A**, **B** and **C**-branes, respectively.

A configuration of (p,q) 7-branes generates an eight-dimensional worldvolume gauge theory, whose gauge group $\mathcal{G}$ is determined by the charges of the 7-branes when their branch cuts are given a well-defined ordering, generalizing the familiar $\mathrm{U}(n)$ gauge group that arises when n 7-branes with the same (p,q) charge coincide. In addition, $\mathcal{G}$ appears as a global symmetry on the worldvolume of some probe brane or branes brought near the 7-brane configuration. Thus in our example, we expect the 7-branes to induce a global symmetry in the five-dimensional theory.

The objects charged under the symmetry of the 7-branes are various strings and string junctions; those beginning and ending on the 7-branes are the adjoint representation, while various other representations appear as string junctions beginning on the 7-branes and ending on another object, such as a 5-brane. Only a (p,q) string can end on a (p,q) 7-brane. String junctions stretch in the plane orthogonal to the 7-branes, and may end on one another in agreement with charge conservation to form webs just as the 5-branes do. The slopes are a ninety-degree rotation from those of the 5-branes, as discussed in section 2.

A given string junction $\mathbf{J}$ with support on n 7-branes is characterized by n invariant charges Q^i, where i labels the 7-branes. The Q^i are integers, and are a combination of the number of string prongs ending on a given brane and the number of string segments passing that brane's branch cut. They measure how much charge flows out of each 7-brane into $\mathbf{J}$, and are invariant under brane creation transformations.

The space of junctions is a lattice, and it has an inner product defined on it, the intersection number. This quantity can be thought of as being inherited from the intersection of curves in K3. (For more details, see [13].) As discussed in [13], the intersection of two junctions $\mathbf{J}$ and $\mathbf{J}'$ receives contributions from each brane on which both have prongs, and from points where a string segment with charges (p,q) belonging to one junction crosses a segment with charges (r,s) belonging to the other:

$$(\mathbf{J},\mathbf{J}') = -\sum_{i=1}^{n} n_i n_i' + \sum_{\text{crossings}} \begin{vmatrix} p & r \\ q & s \end{vmatrix} , \tag{4.1}$$

where n_i is the number of prongs leaving the ith brane minus the number entering it, and the (p,q) and (r,s) segments are ordered in a counterclockwise fashion. This can be expressed entirely in terms of the Q^i and Q'^i, but we shall not do so here.

Alternately, a junction can be specified by its Lie algebra weight vector λ with respect to the group $\mathcal{G}$, which includes possible $u(1)$ charges and is given by $n-2$ Dynkin labels, and the asymptotic charges carried away from the 7-branes, p and q. In terms of these quantities the intersection form is just

$$(\mathbf{J},\mathbf{J}') = -\lambda\cdot\lambda' + f(p,q;p',q') , \tag{4.2}$$

where $\lambda\cdot\lambda'$ is exactly the usual Lie algebraic inner product on weight vectors, and f is a certain expression quadratic in the charges and symmetric between (p,q) and (p',q'), which is specified by the total monodromy of the 7-branes [16]. For the root junctions the asymptotic charges vanish, and the intersection of the simple root junctions is just minus the Cartan matrix of $\mathcal{G}$.

A necessary condition for a junction to be BPS is that it obey [14]

$$(\mathbf{J},\mathbf{J}) \geq -2 + \gcd(p,q) . \tag{4.3}$$

This is because BPS junctions are associated to holomorphic curves in K3, and hence must satisfy $(\mathbf{J},\mathbf{J}) = 2g + b - 2$ where $g \geq 0$ is the genus and $b = \gcd(p,q)$ is the number of boundaries of the curve.

7-brane configurations realizing the simply laced algebras A_n, $D_{n\geq 4}$ and E_6, E_7, E_8 correspond to Kodaira singularities of K3 when the branes are brought to a point. Other brane configurations cannot be so collapsed, such as the $D_{n\leq 3}$ associated to Seiberg-Witten probe theories and $E_{n\leq 5}$. There are distinct 7-brane configurations for E_1 and $\tilde{E}_1$ with SU(2) and U(1) symmetry, respectively, as well as E_0 with no symmetry, exactly as in the 5-brane case.

Perhaps surprisingly, there also exist 7-brane configurations whose associated string junctions fill out the adjoint representation of an infinite-dimensional algebra.

These cases are never collapsible, so the algebra should not be thought of as being a familiar gauge symmetry. In [15], it was shown how the affine versions of all the exceptional algebras, $\widehat{E}_n$, appear when a certain new 7-brane is added to the regular E_n groupings. Interestingly, these are the only affine algebras realized on 7-branes [17].

Returning to the problem of the 7-branes in the 5-brane web, we now ask the question of what symmetry is associated to the 7-branes with the same (p, q) charges as the external 5-brane legs they replaced.

The answer is that for the E_n 5D theory, the full global symmetry of the 7-branes turns out to be $\widehat{E}_n$. Let us return to the example of the E_1 theory. The four associated 7-branes form the configuration **BCBC**. There are two obvious root junctions, the BB and CC strings, which we will call $\boldsymbol{\alpha}_0$ and $\boldsymbol{\alpha}_1$. They have self-intersection $(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_i) = -2$, and intersection $(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1) = 2$, and thus are simple roots for an $\widehat{\mathrm{SU}(2)} = \widehat{E}_1$ algebra.

For the other cases where 5-brane webs realize an E_n theory, the associated 7-branes also have $\widehat{E}_n$ symmetry, even for the cases E_4 and E_5 where there were parallel external legs and the five-dimensional theory was ill-defined. As we shall see, the sequence of 7-branes may be continued all the way up to an $\widehat{E}_8$ configuration, which will realize the E_8 five-dimensional theory.

One might be tempted to conclude from this that the 5D theories should realize an infinite-dimensional global symmetry. However, this proves not to be the case. Before explaining why this is so, let us briefly review a few properties of affine algebras.

Each finite simple compact Lie algebra $\mathcal{G}$ has an affine extension $\widehat{\mathcal{G}}$. If T^a generate $\mathcal{G}$, $\widehat{\mathcal{G}}$ has the generators $\{T^a_n, K, D\}$ where the grade $m \in \mathbb{Z}$, and the commutation relations are

$$\begin{aligned} \left[T^a_{m_1}, T^b_{m_2}\right] &= f^{ab}{}_c\, T^c_{m_1+m_2} + \kappa^{ab}\, m_1\, \delta_{m_1+m_2} K\,, \\ [D, T^a_m] &= -m\, T^a_m\,, \\ [K, T^a_m] &= [K, D] = 0\,, \end{aligned} \tag{4.4}$$

where $f^{ab}{}_c$ and κ^{ab} are the structure constants and Killing form of $\mathcal{G}$, respectively.

Let $\{H^i, E^\alpha\}$ be a set of Cartan and root generators for $\mathcal{G}$. The Cartan subalgebra of $\widehat{\mathcal{G}}$ is $\{H^i_0, K, D\}$. In addition to an infinite number of roots associated to the generators E^α_m, there are also roots associated to H^i_m, $m \neq 0$. These are all integer multiples of the imaginary root δ. The imaginary root has a few interesting properties, including $\delta \cdot \alpha_i = 0$ for any root, including δ itself.

Affine algebras have highest weight representations as finite algebras do, but differ in having no lowest weight. The level $k(\lambda) \equiv \delta \cdot \lambda$ is a constant over a representation, as subtracting simple roots will not change it.

When affine algebras are realized on junctions, the level turns out to be a linear combination of p and q. (Affine configurations are in fact the only ones where the asymptotic charges correspond to a Lie algebraic quantity.) The level can be determined by intersection with the imaginary root junction $\boldsymbol{\delta}$, $k(\mathbf{J}) = -(\mathbf{J}, \boldsymbol{\delta})$. The imaginary root junction obeys $(\boldsymbol{\delta}, \boldsymbol{\delta}) = 0$, and since for a general junction with no asymptotic charges $\mathbf{J}^2 = 2g - 2$, we see that unlike the usual root junctions with $g = 0$, $\boldsymbol{\delta}$ is a genus one object. As such it can be realized as a closed loop surrounding the 7-branes. The (p, q) charge of a segment of the loop changes as it passes each branch cut, but after passing all the branch cuts it comes back to itself, a situation only possible when the 7-branes realize an affine algebra. The intersection is

$$(\mathbf{J}, \mathbf{J}') = -\lambda \cdot \lambda' + f(p, q; p', q') - mk' - m'k\,, \tag{4.5}$$

where $\lambda \cdot \lambda'$ and f are the same as for the configuration realizing the corresponding finite algebra, and m is the grade of the junction, given by the number of $\boldsymbol{\delta}$ factors it contains. For the E_n case we have

$$f(p, q; p', q') = \frac{1}{9-n}\left(pp' - \frac{1}{2}(n-3)(pq' + qp') + (2n-9)qq'\right). \tag{4.6}$$

Let us present $\boldsymbol{\delta}$ for the **BCBC** configuration, which is $\widehat{E}_1$. A $(1, 0)$ string may cross the first **B** branch cut, becoming a $(0, 1)$ string, and then cross the first **C** cut, becoming $(-1, 0)$. The second pair of cuts turn it into $(0, -1)$ and then $(1, 0)$ again, after which it is free to loop around and join itself. Higher $\widehat{E}_n$ configurations are obtained by adding **A**-branes, whose branch cuts do not affect the $(1, 0)$ string, so $\boldsymbol{\delta}$ exists as before. $\widehat{\widetilde{E}}_1$ and E_0 are slightly different, but $\boldsymbol{\delta}$ still starts and ends as a $(1, 0)$ string; details can be found in [15, 17].

Now consider the 5-brane web configuration. Once a 7-brane moves inside the face, the external leg vanishes, and instead it is the 7-brane branch cut that changes the (p, q) character of the 5-brane. When all 7-branes are inside the face, the entire 5-brane configuration now forms a loop around them, changing (p, q) labels at each branch cut but coming back to itself after all branch cuts. In fact, the 5-brane web in each case traces the same curve with the same charges as the imaginary root junction $\boldsymbol{\delta}$.

This is an interesting result. Recall that only affine 7-brane configurations can support nontrivial loops with one face; the affine character is thus essential for the existence of the 5-brane web. This provides an oblique reason why only exceptional affine algebras are realized by 7-brane configurations; others would lead to new five-dimensional theories with other global symmetries, which are not seen in the other string realizations of these theories. In particular, the affine exceptional 7-brane configurations are in one-to-one correspondence with del Pezzo surfaces.

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We have suggested that there should be regions in parameter space where the BPS states are 5-7 strings; this turns out to be a useful way to obtain the quantum numbers of states, since the 7-branes make the global symmetry manifest. However, we must explain why the affine character of the 7-branes is invisible to the field theory.

A junction leaving an $\widehat{E}_n$ configuration of 7-branes is characterized by its finite E_n weight vector λ, a grade m, and charges p and q; one linear combination of these charges determines the level k. However, there is a restriction on the possible quantum numbers of a junction that appears in the 5D theory: its (p,q) charges must match those of the segment of 5-brane on which it ends. But since the 5-brane loop traces the same curve as $\boldsymbol{\delta}$, the level of a junction $\mathbf{J}$ that can end on the 5-brane is, from (4.1):

$$k(\mathbf{J}) \equiv -(\boldsymbol{\delta}, \mathbf{J}) = \begin{vmatrix} p & p \\ q & q \end{vmatrix} = 0\,. \tag{4.7}$$

Thus the restriction on junctions which can end on the 5-brane configuration is precisely that they have vanishing level. This will be the reason why only a finite remnant of the affine algebra survives as a global symmetry in the five-dimensional spacetime.

For definiteness, let us consider junctions that end on the top $(1,0)$ 5-brane; as before, junctions ending elsewhere on the 5-brane loop can be slid along until they end at the top, possibly crossing branch cuts or undergoing brane creation along the way. To end on this 5-brane the junctions must satisfy $q = 0$, and in fact in this presentation $k = -q$ [15]. Now p is related to the electric charge of the state under the Cartan generator of the 5D SU(2) gauge theory n_e [7], by

$$n_e = p/2\,. \tag{4.8}$$

Note that n_e is properly thought of as being a linear combination of the U(1) factors on each 5-brane, which is why it is invariant as the ends of the junction are moved around the inside of the face.

The E_n weight vector λ will characterize the (finite) global symmetry quantum numbers of the state. We are still left with the grade m, which we now claim is irrelevant to the 5D theory.

Consider a junction with p $(1,0)$ prongs on the top D5-brane. Take any one of these prongs and slide it along the loop of 5-brane until it has gone once around and returns to the top. During this process, it must pass through all the branch cuts, and thus its invariant charges change exactly by the addition of the charges of $\boldsymbol{\delta}$; in other words m changes by 1 with other quantities fixed. However, we do not expect any of the field theory quantum numbers to be changed under this process. Hence when a junction ends on a 5-brane loop, the quantity m ceases to be an invariant, but instead can be changed through continuous transformations. The final value of m will be determined simply by the junction minimizing its length. Hence this parameter does not exist in the 5D field theory, consistent with the fact that it has not been observed by other methods. Note how in the junction intersection form (4.5), all dependence on m drops out when $k = 0$, and it reduces to the same intersection form as the finite case (4.2).

A D3-brane in the vicinity of an E_n 7-brane configuration will realize a four-dimensional $\mathcal{N} = 2$ theory with E_n global symmetry. The junctions leaving the 7-branes with $q = 0$ are in one-to-one correspondence with the ones realizing the BPS spectrum from $\widehat{E}_n$ configurations in the five-dimensional case; thus the electric spectrum in the four-dimensional theory is the same as that in the five-dimensional theory. This is as we would expect from the del Pezzo picture, since both come from wrapping 2-branes on 2-cycles in the del Pezzo, either in M-Theory or Type IIA string theory.

5. E1 states from junctions

Let us see in detail how the BPS states of the E_1 theory are realized as junctions.

The basic states are the W-boson and the instanton, which we converted into 5-7 junctions in fig. 4. We can read off the invariant charges Q^i and obtain the quantum numbers using the techniques of [13, 15]:

$$\begin{aligned} p &= Q_{B1} + Q_{B2} + Q_{C1} + Q_{C2}\,, \\ a &= Q_{C1} - Q_{C2} + 2Q_{B2}\,. \end{aligned} \tag{5.1}$$

We find

$$\begin{aligned} &\text{W-boson:} \qquad p = 2\,, \qquad a = 1\,, \\ &\text{Instanton:} \qquad p = 2\,, \qquad a = -1\,, \end{aligned} \tag{5.2}$$

where a is the single $E_1 = \mathrm{SU}(2)$ Dynkin label characterizing λ. We see both have $n_e = 1$, and that they fall into a doublet of the global symmetry, exactly as we expect. Both junctions have $\mathbf{J}^2 = 0$, with $g = 0$ and $b = 2$. The instanton number used in [7] is given by $I = \frac{1}{2}(p/2 - a)$.

Other BPS states are linear combinations of these two. Under such combinations, the invariant charges add, and since quantities p and a are linear in the invariant charges, they will also add. Hence the charges and global symmetry representations of arbitrary BPS states can be deduced easily from the junction perspective.

Which junctions $\mathbf{J}$ are permitted as BPS states? The most important constraint is (4.3), which becomes

$$(\mathbf{J}, \mathbf{J}) = -\frac{1}{2}a^2 + \frac{1}{8}p^2 \geq -2 + p\,. \tag{5.3}$$

In terms of the five-dimensional parameters n_e and I, this is

$$n_e(I-1) \geq (I+1)(I-1)\,, \tag{5.4}$$

and the set of permitted states precisely coincides with those given by [9]. In addition, here we have shown definitively that bound states of m W-bosons, $m > 1$, with no instantons are not BPS, without having to appeal to reducibility.

It should not be too surprising that the results coincide. The spectrum was determined in [9] using the intersection numbers of curves in del Pezzo surfaces, which we will describe further shortly. Our results come ultimately from intersection numbers determined by curves in K3. Once the basis curves are identified in each geometrical setup, and their intersection numbers shown to agree, the inner product on both homology lattices agrees and so the constraints should coincide. However, [9] used a mutual intersection condition, while we employ a self-intersection condition. In other cases (such as $\tilde{E}_1$) we must supplement the self-intersection constraint with a mutual intersection constraint. This is described in the next section.

6. Del Pezzo surfaces and BPS spectra

The five-dimensional theories we have been studying can be obtained by compactifying M-Theory on a Calabi-Yau threefold containing a shrinking four-cycle called a del Pezzo surface. The E_0 theory is obtained when the del Pezzo is just $\mathbb{P}^2$, and the $\tilde{E}_1$, $E_{2\leq n\leq 8}$ theories correspond to $\mathbb{P}^2$ blown up at n generic points, denoted $\mathcal{B}_n$. Finally, for E_1 the four-cycle is $\mathbb{P}^1 \times \mathbb{P}^1$. The BPS particle states in the field theory are obtained by wrapping M2-branes on various 2-cycles within the del Pezzo. Hence the second homology lattice of del Pezzo surfaces determines the BPS spectrum. The intersection of these curves coincides with the intersection of the junctions associated to the same BPS states.

An $n+1$-dimensional basis for the 2-cycles in $\mathcal{B}_n$ is $\{\ell, e_1, \ldots, e_n\}$ where ℓ is a $\mathbb{P}^1$ inside $\mathbb{P}^2$ and the e_i are the exceptional divisors of the blown-up points. The intersection matrix in this basis is diag $(1, -1, \ldots, -1)$ and the canonical class is given by $K_{\mathcal{B}_n} = -3\ell + \sum_{i=1}^n e_i$. For $\mathbb{P}^1 \times \mathbb{P}^1$ the basis is ℓ_1 and ℓ_2, with $\ell_1 \cdot \ell_1 = \ell_2 \cdot \ell_2 = 0$, $\ell_1 \cdot \ell_2 = 1$, and canonical class $K_{P^1 \times P^1} = -2(\ell_1 + \ell_2)$. We denote a general del Pezzo by X.

The degree d_C of a curve C is defined by intersection with the canonical class K_X,

$$d_C = -C \cdot K_X\,. \tag{6.1}$$

We will identify d with the electric charge of the corresponding state, $d = p = 2n_e$.

In each case the homology lattice contains the root lattice of the corresponding E_n algebra. The set of roots is defined as curves with vanishing degree and self-intersection -2,

$$\{\alpha_i\} = \{C \in H^{1,1}(X) | (C \cdot C) = -2, (C \cdot K_X) = 0\}\,, \tag{6.2}$$

and as such they generate a degree-preserving E_n Weyl group action on all curves.

Explicitly the simple roots for E_n, $\{3 \leq n \leq 8\}$ are

$$\alpha_i = e_i - e_{i+1}\,, \qquad i = 1, \ldots, n-1 \qquad \text{and} \qquad \alpha_n = \ell - e_1 - e_2 - e_3\,. \tag{6.3}$$

For E_2 the SU(2) root is $e_1 - e_2$ and for E_1 it is $\ell_1 - \ell_2$. The Dynkin labels a_i of a curve characterizing its E_n weight λ are given by

$$a_i = -C \cdot \alpha_i\,, \qquad i = 1, \ldots, n\,. \tag{6.4}$$

In the E_2 and $\tilde{E}_1$ cases the $u(1)$ factor is not associated with a root, but a curve not satisfying (6.2) can be chosen to give the appropriate generalized Dynkin label. An arbitrary curve C thus can be specified by its n Dynkin labels and its degree d, in terms of which the intersection of two curves C and C' is

$$(C \cdot C') = -\lambda \cdot \lambda' + \frac{d\,d'}{9-n}\,. \tag{6.5}$$

With the identification $p \leftrightarrow d$, this coincides exactly with the inner product on junctions given in (4.5) with f given in (4.6) and $k = -q = 0$. Thus with each BPS state in an E_n five-dimensional theory, we can associate both a junction on a $\widehat{E}_n$ set of 7-branes and a holomorphic curve in the corresponding del Pezzo surface.

6.1 Mutual intersection constraint

Unlike in $K3$, in a del Pezzo surface X self-intersection number alone is not sufficient to determine if a given homology class has a holomorphic representative or not. Holomorphic curves must satisfy the adjunction formula:

$$C \cdot C + C \cdot K_X = 2g - 2\,, \tag{6.6}$$

which corresponds to (4.3), with d substituted for $\gcd(p, 0) = p$. Since K3 is Calabi-Yau and its canonical class vanishes, a corresponding expression (6.6) would have no term $C \cdot K$, but it is compensated for exactly by the inclusion of curves with boundary components.

In addition, del Pezzo surfaces have a mutual intersection constraint: holomorphic curves have positive degree and are required to have positive intersection with all curves of degree one and self-intersection number minus one [5, 25]. (Curves of negative degree can be antiholomorphic, and are just the negative of the holomorphic curves.) We will now derive a single constraint ensuring this requirement, simplifying the multiple constraints of [5].

Curves with $C \cdot C = -1$ satisfying (6.6) with positive degree are necessarily $g = 0$ and $d = 1$. By (6.5), they fall into a representation $R(n)$ of E_n with

$$\lambda^2 = \frac{10-n}{9-n}\,. \tag{6.7}$$

For $E_{3\leq n\leq 8}$ these representations are $(\mathbf{3},\mathbf{2}), \mathbf{10}, \mathbf{16}, \mathbf{27}, \mathbf{56}, \mathbf{248}$. They can be thought of as the fundamental representations of E_n. No such curves exist for E_0 and E_1 theories, so the mutual intersection constraint is trivial. For $\tilde{E}_1$ and E_2 there are U(1) factors rendering the definition of λ conventional; we shall consider these theories individually.

In general it is tedious to check the positivity of the intersection of some curve $C = (\lambda, d)$ with every curve in $R(n)$. However, we can simplify the task as follows. We wish to ascertain

$$(C, C_i) = -\lambda \cdot \lambda_i + \frac{d}{9-n} \geq 0\,, \tag{6.8}$$

for all $\lambda_i \in R(n)$, or in other words,

$$d \geq (9-n)\lambda \cdot \lambda_i\,. \tag{6.9}$$

Since the multiplication of weights is just the Euclidean inner product, the most stringent condition arises when λ and λ_i are in the same Weyl chamber. By a Weyl rotation we can make this the fundamental Weyl chamber, and thereby ascertain (6.8) for all λ in a given representation R by checking the intersection of the highest weight $\lambda_h(R)$ with just the highest weight of $R(n)$:

$$d \geq (9-n)\,\lambda_h(R) \cdot \lambda_h(R(n))\,. \tag{6.10}$$

One must be careful to notice that for the curves with $g = 0$, $d = 1$ themselves, the mutual intersection constraint becomes self-intersection, which is less stringent. For example for the case $n = 8$, $R = \mathbf{248}$ (6.8) seems to imply we need $d \geq 2$, but in fact since $R = R(8)$ itself this is not a mutual intersection constraint, and we can have $d \geq 1$.

We can translate this directly into a mutual-intersection constraint to be imposed on string junctions $\mathbf{J} = (\lambda, p)$ in the (p, q) 5-brane/7-brane setup:

$$p \geq \lambda_h(R) \cdot \lambda_h(R(n))\,. \tag{6.11}$$

6.2 Relating D7-branes to geometrical blow-ups

In this section we will relate the geometric picture with the brane picture by establishing a correspondence between blowing up points and the addition of D7-branes.

The del Pezzos which admit a description in toric geometry are the ones associated to E_0, $\tilde{E}_1$, E_1, E_2 and E_3, precisely those theories which can be realized as a (p,q) 5-brane web without 7-branes. Consider $\mathbb{P}^2$ and its one-dimensional homology lattice generated by ℓ with $\ell^2 = 1$. This manifold admits a toric description, with a toric skeleton looking exactly like the 5-brane web, fig. 5a. We can blow up a generic point of $\mathbb{P}^2$ and obtain the $\mathcal{B}_1$ del Pezzo surface, introducing the new 2-cycle e, with $e^2 = -1$, $\ell \cdot e = 0$, as fig. 5b.

Notice to obtain $\mathcal{B}_1$ we blew up a point on the boundary of the face. This is not a special point on $\mathbb{P}^2$, only in our toric description of it. It is not possible to represent torically a blown up generic point inside the face; only the corners of the toric diagram of $\mathbb{P}^2$ can be blown up by replacing the point with a $\mathbb{P}^1$ which would be represented by a line segment in the toric diagram. But due to an $\mathrm{SL}(3,\mathbb{C})$ symmetry of $\mathbb{P}^2$ any point can be mapped to the point on the corner, and thus a generic blow-up of $\mathbb{P}^2$ can be represented by a toric diagram after this $\mathrm{SL}(3,\mathbb{C})$ transformation. However, this process cannot be continued indefinitely since the $\mathrm{SL}(3,\mathbb{C})$ symmetry is exhausted after three blow-ups. Therefore there is no toric description of $\mathbb{P}^2$ blown up at more than three points; this creates the familiar parallel external legs in the toric skeleton, which in a toric description means the manifold no longer can go to zero size inside the Calabi-Yau [10].

Adding 7-branes to the 5-brane web, we can avoid this problem, effectively blowing up points on the interior of the face and constructing all the E_n theories.

We claim that adding a D7-brane in the interior of the 5-brane web corresponds to blowing up a generic point on the del Pezzo. Geometrically, a blow-up introduces

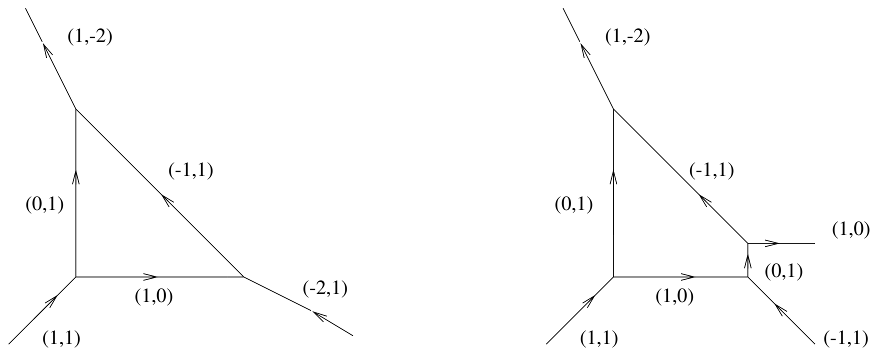

Figure 5: a) The toric skeleton for $\mathbb{P}^2$, which gives the E_0 theory; the slopes are the degenerating cycles. This is identical to the corresponding (p,q) 5-brane web. b) The toric skeleton for $\mathcal{B}_1$, $\mathbb{P}^2$ blown up at a generic point, here chosen to be on the boundary.

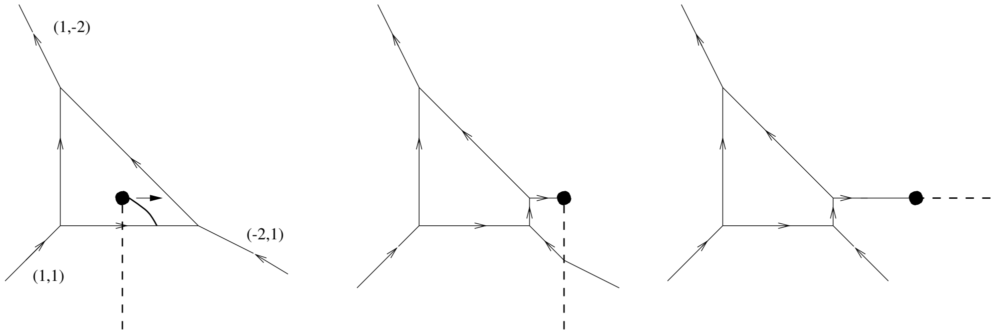

Figure 6: Adding a D7-brane to the (p, q) 5-brane web of E_0 corresponds to blowing it up to $\tilde{E}_1$ at an interior point, with a new flavor appearing as a string ending on the D7; moving the D7-brane outside the face returns us to the usual blow-up at the edge.

a genus zero homology cycle of self-intersection minus one. Correspondingly, adding a D7-brane introduces a new dimension to the junction lattice, generated by the string leaving the D7-brane, which indeed has genus zero and self-intersection -1. Moving this D7-brane outside the 5-brane web generates an external D5-brane (fig. 6) and returns to the picture with the blow-up on the boundary of the toric diagram. However, we are now free to add more D7-branes beyond the point where we are able to bring them outside the face, corresponding to blowing up points beyond $\mathcal{B}_3$. We can continue to add D7-branes inside the face all the way to E_8, thus solving the problem of a brane description of these theories.

A more precise geometrical interpretation of a general (p, q) 7-brane in the toric picture is still unknown, and would be very interesting to determine. The obstruction to proceeding beyond E_8 will be explored in section 7.

6.3 BPS spectra for E_n theories

Here we summarize the BPS spectra for all the five-dimensional E_n theories. For E_0, E_1 and $\tilde{E}_1$ this has already been discussed in [9]. Related work on E_6, E_7 and E_8 spectra can be found in [5]. We will review the results for the first few theories, and then concentrate on $E_{n\geq 2}$.

As discussed previously, there is no additional mutual intersection constraint for E_0 and E_1, and the self-intersection constraint (4.3) suffices. The basic BPS state of the E_0 theory is the state Δ with $n_e = 3/2$, corresponding to the cycle ℓ. All states $m\,\Delta$ for integer m are permitted by (4.3). The lack of a W-boson demonstrates this is not a gauge theory.

E_1 is pure SU(2) with $\theta = 0$, and the del Pezzo is $\mathbb{P}^1 \times \mathbb{P}^1$. The basic states are the W-boson and the instanton, corresponding to the two curves ℓ_1 and ℓ_2, each with $n_e = 1$ and the latter with unit instanton number I. The self-intersection constraint evaluates to (5.4), permitting any state $n\ell_1 + m\ell_2$ with n, m the same sign, except when either integer vanishes, in which case the other must be ± 1.

$\tilde{E}_1$ is the pure SU(2) theory with $\theta = \pi$, but the del Pezzo is that of the E_0 theory blown up at a point. Δ appears as the curve ℓ, and the curve e arises, the associated state being the "instantonic quark" I_Q with $n_e = 1/2$; the W-boson is the linear combination $\ell - e$. One characterizes the states by n_e and the instanton number I, with $I = 0$ for the W-boson and $I = 1$ for Δ, I_Q. The mutual intersection constraint with the single curve e, combined with the self-intersection constraint, forces all BPS states to be of the form $n\ell + m(\ell - e)$ with n, m of the same sign, with the exception of e itself, which is also BPS, and the states with $n = 0$, which must have $m = \pm 1$.

In [9], the identical $\tilde{E}_1$ spectrum was obtained by imposing two mutual intersection constraints, positivity of intersection with e and in addition with $\ell - e$, and no self-intersection constraint; the states $n = 0$, $|m| > 1$ were then argued not to be BPS states since they are reducible. It is gratifying that we can reproduce these results with our methods, which apply equally well to every E_n theory, and which need not be supplemented by appeals to reducibility.

States in the $E_{n\geq 2}$ theories are characterized by electric charge n_e, instanton number I, and $n-1$ global symmetry charges associated to quarks, Q_{f_i}. These should correspond to junction/del Pezzo quantities. We have stated that $p = n_e/2$; the instanton number I and the Q_{f_i} will be determined by the weight vector λ. Below we consider the constraints on the $n_e > 0$ states, which correspond to holomorphic curves; the $n_e < 0$ ones are just the negatives, corresponding to antiholomorphic curves.

Specifically, it is easy to see from fig. 6 that the quarks all end on the D7-brane that adds the flavor. Thus, the quark charge Q_{f_i} is just the invariant charge Q_{Ai} on the appropriate D7-(or **A**-)brane:

$$Q_{f_i} = Q_{Ai}\,. \tag{6.12}$$

For E_1 we had $I = \frac{1}{2}(p/2 - a)$ where a was the SU(2) Dynkin label. We can think of all junctions in $E_{n\geq 2}$ theories as the sum of an E_1 junction $\mathbf{J}_{E_1}$ with some p' and a, thus determining I, and various strings from the **A**-branes, determining Q_{f_i}: the total charge is then $n_e = p/2 = (p' + \sum_i Q_{f_i})/2$. The total self-intersection is given by (4.1)

$$\mathbf{J}^2 = -\frac{1}{2}a^2 + \frac{1}{8}(p')^2 - \sum_i (Q_{f_i})^2\,, \tag{6.13}$$

where the determinants vanish since the asymptotic strings are mutually local. Then in terms of n_e, I and Q_{f_i} the self-intersection constraint (4.3) is

$$I\left(2n_e - 2I - \sum_i Q_{f_i}\right) - \sum_i (Q_{f_i})^2 \geq 2(n_e - 1)\,, \tag{6.14}$$

or

$$2n_e(I-1) \geq 2(I+1)(I-1) + I\sum_i Q_{f_i} + \sum_i (Q_{f_i})^2\,. \tag{6.15}$$

The mutual intersection constraint is most easily evaluated for an entire representation. To this end, we provide the explicit map between I, Q_{f_i} and λ, and then list the constraint on the weight vector.

For $E_{n\geq 3}$, we have

$$\begin{aligned} a_1 &= n_e - 2I - \frac{1}{2}\sum_i Q_{f_i}\,, \\ a_2 &= Q_{f_1} + Q_{f_2} + I\,, \\ a_k &= Q_{f_{k-1}} - Q_{f_{k-2}}\,, \qquad k = 3, \ldots, n\,. \end{aligned} \tag{6.16}$$

For all these cases, the mutual intersection constraint (6.11) becomes

$$p \geq 2a_1 + 4a_2 + 3a_3 + 6a_4 + 5a_5 + 4a_6 + 3a_7 + 2a_8\,, \tag{6.17}$$

where for $n < 8$ one merely ignores the terms with a_i, $i > n$. These numbers are exactly the Coxeter labels of E_8; for the other E_n the truncated set is the highest weight of $R(n)$ expanded in the basis of simple roots. Plugging in (6.16), we obtain that the highest weight must satisfy

$$Q_{f_{n-1}} \leq 0\,. \tag{6.18}$$

The result (6.18) is transparent from the junction point of view, using (4.1) and the fact that the highest weight state of $R(n)$ is simply the open string beginning on the last D7-brane.

For E_2, we have

$$\begin{aligned} a_1 &= n_e - 2I - \frac{1}{2}Q_f\,, \\ q &= 2Q_f + I\,, \end{aligned} \tag{6.19}$$

where q is the U(1) charge. However the U(1) factor destroys our ability to classify the states by the Weyl orbits of a root system. Instead we give the allowed states explicitly. The basis states are the W-boson ℓ_1 with $n_e = 1$, the instanton ℓ_2 with $n_e = 1$ and $I = 1$, and the quark e with $n_e = \frac{1}{2}$ and $Q_f = 1$.

The self-intersection constraint is

$$2n_e(I-1) \geq 2(I+1)(I-1) + IQ_f + (Q_f)^2\,, \tag{6.20}$$

and mutual intersection constrains $I \geq 0$, $Q_f \leq 0$, and $2n_e <\geq 2I + Q_f$; the negative of any state is also BPS.

7. Approaching the fixed point and global symmetry

Thus far, we have described how one may use 7-branes to learn about the global symmetry representations of the 5D states. However, we have not commented on how these symmetries are realized in the 5D theory. We recall that symmetry enhancement occurs only when 7-branes are coincident; this is reminiscent of the situation in [21], where symmetry enhancement was only explicit from the string point of view when the D5-branes were coincident. In the K3 language, coincident branes correspond to colliding degenerate fibers forming a singularity. The types of possible deformable singularities where classified by Kodaira. Since it is clear that affine groups do not appear in Kodaira's classification, none of the 7-brane configurations we have considered are fully collapsible [17]. On the other hand, at the conformal fixed point theory no distance scales should be present.

To solve this puzzle let us return to the E_1 configuration of fig. 3. The 7-branes may move along certain geodesics, corresponding to the path that was followed by the external 5-brane legs as modified by the nontrivial metric, without affecting the 5D theory. Hence, we may attempt to put the 7-branes on top of one another by sliding them along these geodesics. By inspecting the configuration in fig. 3, it is clear that 7-branes may encounter each other only at infinite coupling, when the parameters are tuned such that the face is a square. We may choose the two **C**-branes to be coincident, and ask what happens to the remaining **B**-branes.

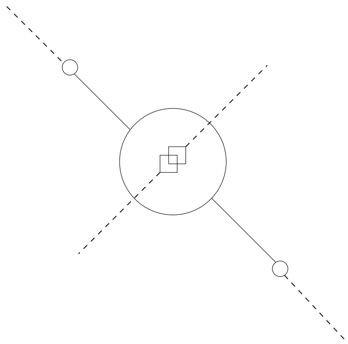

Figure 7: As the E_1 system approaches the fixed point, only two 7-branes can collapse, restoring the global symmetry.

Since **BCC** is not collapsible, once the two **C**-branes are placed in middle of the face, the other branes must always remain a finite distance away. As the modulus is decreased, with the face closing towards the fixed point, 5-brane prongs will inevitably form on the **B**-branes (fig. 7). The BPS states of the 5D theory will either be 5-5 strings or 5-7 strings emanating from the **C**-branes. When the face closes, all these become massless. On the other hand, the **B**-branes seem superficially to provide a distance scale, spoiling the conformal fixed point. However, since the theory (as before) is insensitive to the position of the **B**-branes along their geodesics, the 5D theory is ignorant of this scale.

This is a crucial point for product group symmetry, as in E_3 for instance. There, the global SU(2) $\times$ SU(3) is achieved by 2 branes of one type coinciding and 3 branes of another type coinciding (happening only at infinite coupling and zero bare masses). The two types are mutually non-local and non-collapsible. Therefore, it is not possible for both sets of coincident branes to be in the center of the face. Instead, the face will close in between the two sets, with external 5-brane legs on the two sets of branes. Again, the 5D theory's ignorance of the distance along the external legs ensures that the symmetry is in fact the full product, rather than just one of its components. Because the legs may be extended to infinity, the fixed point picture is essentially the original 5-brane web without 7-branes. More generally, since there are no product groups in Kodaira's classification, we see that a product group fixed point is possible, only if there is a 5-brane web without 7-branes to realize it.

As discussed, the higher E_n series ($n \geq 4$) have no description solely in terms of 5-branes, and as a result as the theory approaches the origin of the Coulomb branch we do not expect the 5-7 brane system to revert to the 5-brane web without 7-branes. Instead, the global symmetry must originate from a loop closing on a fully collapsible portion of the original affine configuration. In particular, this implies that there should not be product groups among the higher E_n series, which is the case. Indeed, all $n \geq 4$ configurations may be represented by a single set of collapsible branes responsible for the symmetry, and a few remaining branes; as this is somewhat technical we present it in an appendix. These remaining branes, although essential to allowing a closed 5-brane loop (and thus a 5D theory) to exist, play no role in the symmetry. For these cases, the fixed point picture has the face closing upon a single point — the location of the global symmetry branes. The remaining branes will be connected by external legs to the 5-brane loop. This picture makes it manifest that the states becoming massless indeed transform under the expected global symmetry.

In addition, one may use the above picture to understand why the maximum number of flavors is 7. Adding flavors is achieved by adding **A**-branes to the E_1 configuration. First, we must ask if there are restrictions in constructing a 7-brane background with any number of **A**-branes. Asymptotically, the monodromy around an affine configuration prescribes the complex parameter to be

$$\tau(z) = \frac{i}{2\pi}(8-n)\log(z) + \tau_0 \,. \qquad (7.1)$$

Here, n is the number of **A**-branes, $|z| > 1$, and τ_0 is a constant. The metric around the 7-branes is proportional to $\mathrm{Im}\tau$, which therefore must always be positive. This requires $n \leq 8$.

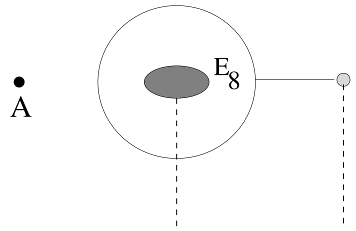

Figure 8: A configuration with 8 flavors is not associated with a new fixed point, as an E_9 configuration of 7-branes cannot collapse, and a scale associated to a D7-brane (A) remains.

The case of $n = 8$ is more subtle. This is the so-called $\widehat{E}_9$ configuration, realizing a doubly-affinized E_8 algebra [15, 17], with trivial monodromy and the ability to support a string loop of any charge winding around the configuration. Only the E_8 part is collapsible; in addition there is the affinizing brane that made $\widehat{E}_8$ and the **A**-brane associated to the eighth flavor that made $\widehat{E}_9$, and these two branes stay a finite distance from the E_8. As the face shrinks upon the collapsed E_8 point, the loop will form an external leg on the affinizing brane as in the 7-flavor (E_8) case, but will completely ignore the **A**-brane, since they are mutually local. Thus, the distance of the **A**-brane, corresponding to the mass parameter of the eighth flavor, stays as a scale in the 5D theory and can never be set to zero. One can only reach a conformal fixed point by integrating out this flavor, thus returning to the E_8 case.

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A. 7-brane technology

In this section we will review some of the properties of the 7-brane configurations realizing the $\widehat{E}_n$ symmetry. We will show that it is always possible to find a sub-configuration of 7-branes realizing E_n symmetry which can be collapsed.

A $[p,q]$ 7-brane has an $\mathrm{SL}(2,\mathbb{Z})$ monodromy matrix given by

$$K_{[p,q]} = \begin{pmatrix} 1+pq & -p^2 \\ q^2 & 1-pq \end{pmatrix}. \tag{A.1}$$

An $\binom{r}{s}$-string crossing the branch cut of the $[p,q]$ 7-brane is converted into an $K_{[p,q]}\binom{r}{s}$-string.

The canonical presentation of 7-brane configurations is the one in which the branes are located along the real axis with branch cuts going downwards. This is convenient for discussing symmetry properties. With this convention the total monodromy around the brane configuration

$$\mathbf{X}_{[\mathbf{p_1},\mathbf{q_1}]}\ldots\mathbf{X}_{[\mathbf{p_{n-1}},\mathbf{q_{n-1}}]}\mathbf{X}_{[\mathbf{p_n},\mathbf{q_n}]}\,, \tag{A.2}$$

is given by

$$K = K_{[p_n,q_n]}K_{[p_{n-1},q_{n-1}]}\ldots K_{[p_1,q_1]}. \tag{A.3}$$

Because of the $\mathrm{SL}(2,\mathbb{Z})$ symmetry of Type IIB string theory and the arbitrariness of the location of branch cuts, two configurations of 7-branes are *equivalent* if they can be transformed into each other via $\mathrm{SL}(2,\mathbb{Z})$ transformations and branch cut moves. Equivalent configurations naturally realize the same algebra.

Global transformation. The monodromy matrix K of a 7-brane can also be expressed in a charge-vector notation. If $\mathbf{z}^T \equiv [p,q]$ is the 7-brane charges,

$$K_{\mathbf{z}} = 1 + \mathbf{z}^T \mathbf{z}\, S\,, \qquad \text{where} \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{A.4}$$

Under a global transformation $g \in \mathrm{SL}(2,\mathbb{Z})$, we have $\mathbf{z} \to g\,\mathbf{z}$. It is easy to see from (A.4) that the monodromy undergoes a conjugation by g,

$$K_{g\mathbf{z}} = g K_{\mathbf{z}} g^{-1} \tag{A.5}$$

Branch cut moves. The labeling of branes actually depends on the placement of branch cuts. We can move the branch cut of one 7-brane $\mathbf{X}_{\mathbf{z}_1}$ across another 7-brane $\mathbf{X}_{\mathbf{z}_2}$, thus changing the latter to $\mathbf{X}_{\mathbf{z}'_2}$ and exchanging their order in the canonical presentation, as explained in [12], and in the canonical presentation they become

$$\begin{aligned} \mathbf{X}_{\mathbf{Z}_1}\mathbf{X}_{\mathbf{Z}_2} &= \mathbf{X}_{\mathbf{Z}_2}\,\mathbf{X}_{(\mathbf{z}_1 + (\mathbf{z}_1 \times \mathbf{z}_2)\,\mathbf{z}_2\,)} \\ &= \mathbf{X}_{(\mathbf{z}_2 + (\mathbf{z}_1 \times \mathbf{z}_2)\,\mathbf{z}_1\,)}\,\mathbf{X}_{\mathbf{Z}_1}\,, \end{aligned} \tag{A.6}$$

where we have defined

$$\mathbf{z}_1 \times \mathbf{z}_2 \equiv -\mathbf{z}_1^T S\, \mathbf{z}_2 = \mathbf{z}_2^T S\, \mathbf{z}_1 = \det \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix}. \tag{A.7}$$

Equation (A.6) indicates the fixed brane acquires an extra charge equal to the charge of the moving brane times the determinant of the relative charges.

The 7-brane configurations associated with E_n symmetry will be denoted by $\mathbf{E_n}$. One useful presentation of $\mathbf{E_n}$ is $\mathbf{A^{n-1}BCC}$ for $n > 0$ [11]. The intersection of junctions supported on $\mathbf{E_n}$ is given by (4.2) with $f(p,q;p',q')$ given in (4.6).

For $n \geq 2$, the configuration $\mathbf{A^{n-1}BCC}$ is equivalent to $\mathbf{A^n X_{[2,-1]} C}$ [16]. For $n = 1$ the equivalence does not hold; $\mathbf{E_1} = \mathbf{BCC}$ and $\tilde{\mathbf{E}}_\mathbf{1} = \mathbf{AX_{[2,-1]}C}$ are distinct configurations. Furthermore we have $\mathbf{E_0} = \mathbf{X_{[2,-1]}C}$ which can be "blown up" only to $\tilde{\mathbf{E}}_\mathbf{1}$ by adding an $\mathbf{A}$-brane.

The $\mathbf{E_n}$ configuration can be "affinized" by adding another 7-brane to obtain $\widehat{E}_n$ symmetry:

$$\widehat{\mathbf{E}}_\mathbf{n} = \mathbf{A}^{n-1}\mathbf{BCCX_{[3,1]}} = \mathbf{A}^{n-1}\mathbf{BCBC}\,. \tag{A.8}$$

The affinizing brane can never be collapsed onto the others, so the $\widehat{E}_n$ symmetry cannot be restored.

The affinizing brane is a spectator in the equivalence between the two presentations of E_n, so we can analogously construct $\widehat{\tilde{\mathbf{E}}}_\mathbf{1} = \mathbf{A^1 X_{[2,-1]} C X_{[4,1]}}$ and $\widehat{\mathbf{E}}_\mathbf{0} = \mathbf{X_{[2,-1]} C X_{[4,1]}} = \mathbf{X_{[2,-1]} X_{[-1,2]} C}$.

We summarize the correspondence of these brane configurations with the geometry of del Pezzos in the following table.

5D Field Theory	Brane Configuration	Geometry
E_0	$\mathbf{E_0} = \mathbf{X_{[2,-1]}X_{[-1,2]}C}$	$\mathbb{P}^2$
$\tilde{E}_1$	$\mathbf{AX_{[2,-1]}X_{[-1,2]}C}$	$\mathcal{B}_1$
E_1	$\mathbf{BCBC}$	$\mathbb{P}^1 \times \mathbb{P}^1$
$E_n, n \geq 2$	$\mathbf{A}^{n-1}\mathbf{BCBC} = \mathbf{A}^n\mathbf{X_{[2,-1]}CX_{[-1,2]}}$	$\mathcal{B}_n$

Notice that while the 7-brane configurations for E_n as given are natural from the point of view of "blowing-up" by adding **A**-branes as discussed in section 6, some equivalent configuration may be natural when the 7-branes are moved outside the 5-branes.

In section 7, we stated that for each $\widehat{\mathbf{E}}_\mathbf{n}$ there existed a sub-configuration of collapsible 7-branes realizing the finite symmetry E_n. We are now in a position to describe these explicitly. An equals sign denotes a branch cut move, while a $\cong$ with an $\mathrm{SL}(2,\mathbb{Z})$ matrix above it denotes a global $\mathrm{SL}(2,\mathbb{Z})$.

For $\widehat{\mathbf{E}}_\mathbf{6}, \widehat{\mathbf{E}}_\mathbf{7}$ and $\widehat{\mathbf{E}}_\mathbf{8}$, the full finite subconfigurations $\mathbf{E_6}$, $\mathbf{E_7}$, and $\mathbf{E_8}$ correspond to Kodaira singularities and are collapsible.

For $\widehat{\mathbf{E}}_\mathbf{5}$ we expect a collapsible sub-configuration realizing $E_5 = D_5$. The usual presentation of 7-branes realizing D_5 is $\mathbf{A^5BC} = \mathbf{D_5}$. We can see from the following branch cut moves and global $\mathrm{SL}(2,\mathbb{Z})$ that $\mathbf{D_5}$ is indeed present in $\widehat{\mathbf{E}}_\mathbf{5}$.

$$\begin{aligned}\mathbf{A^4BCBC} &= \mathbf{A^5X_{[2,-1]}CX_{[4,1]}} \overset{T^{-2}}{\cong} \mathbf{A^5X_{[4,-1]}BX_{[2,1]}} \\ &= \mathbf{CA^5BX_{[2,1]}} = (\mathbf{X_{[0,1]}})^5\mathbf{CBX_{[2,1]}} \overset{S}{\cong} \mathbf{A^5BCX_{[-1,2]}}\,. \end{aligned} \tag{A.9}$$

For $\widehat{\mathbf{E}}_\mathbf{4}$ we expect a sub-configuration realizing A_4. The usual presentation of A_4 is $\mathbf{A^5}$. The following branch cut moves and global $\mathrm{SL}(2,\mathbb{Z})$ makes it clear that such a subconfiguration is indeed present in $\widehat{\mathbf{E}}_\mathbf{4}$:

$$\begin{aligned}\mathbf{A^3BCBC} &= \mathbf{A^4X_{[2,-1]}CX_{[4,1]}} = \mathbf{A^4X_{[2,-1]}X_{[-1,2]}C} = \mathbf{A^4X_{[5,-1]}X_{[2,-1]}C} = \\ &= \mathbf{BA^4X_{[2,-1]}C} = \mathbf{BX_{[2,-1]}B^4C} = \mathbf{B^5X_{[-2,3]}C} \overset{g}{\cong} \mathbf{A^5X_{[-2,1]}X_{[1,2]}}. \end{aligned} \tag{A.10}$$

with $g = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. For $\widehat{\mathbf{E}}_\mathbf{3}$ we have,

$$\begin{aligned}\mathbf{A^2BCBC} &= \mathbf{A^3X_{[2,-1]}CX_{[4,1]}} = \mathbf{A^3X_{[2,-1]}X_{[-1,2]}C} \overset{T}{\cong} \mathbf{A^3BX_{[1,2]}X_{[2,1]}} = \\ &= \mathbf{B}(\mathbf{X_{[0,1]}})^3\mathbf{X_{[1,2]}X_{[2,1]}} = \mathbf{B^2}(\mathbf{X_{[0,1]}})^3\mathbf{X_{[2,1]}} \overset{g'}{\cong} \mathbf{A^2}(\mathbf{X_{[0,1]}})^3\mathbf{X_{[2,3]}}\,, \end{aligned} \tag{A.11}$$

with $g' = \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$. From the above presentation of $\widehat{\mathbf{E}}_\mathbf{3}$ we see that it has a collapsible sub-configuration realizing $E_3 = A_1 \times A_2$.

For $\widehat{\mathbf{E}}_\mathbf{1}$ and $\widehat{\mathbf{E}}_\mathbf{2}$ there is a manifest $\mathrm{SU}(2)$ when two **C**-branes come together, as in section 7. The $\mathrm{U}(1)$ factors in $\widehat{\mathbf{E}}_\mathbf{2}$ and $\widehat{\tilde{\mathbf{E}}}_\mathbf{1}$ are not associated with collapsing branes.

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